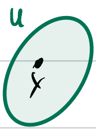
Metric Spaces and Topology
Lecture 5

Def. In a metric space (X, d), a sequence (x_n) is said to **converge** to $x \in X$ if $\forall$ neighbourhood U of x, $\forall^\infty n$ $x_n \in U$. We denote this by $\lim_n x_n = x$ or $x_n \xrightarrow[n\to\infty]{} x$.



Obs. For a sequence (x_n) and $x \in X$, TFAE (The Following Are Equivalent):

(a) $x_n \to x$.

(b) $\forall B_\varepsilon(x)$ $\forall^\infty n \in \mathbb{N}$ $x_n \in B_\varepsilon(x)$ (replace neighbourhood by ball).

(c) $d(x_n, x) \to 0$.

Uniqueness of limit. The limit is unique in metric spaces.

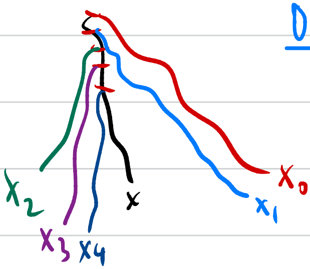
Proof. Let $x_n \to x$ and $x_n \to x'$. It's enough to show that $d(x', x) < \varepsilon$ for each $\varepsilon > 0$ (the "give yourself ε of room" trick). By Δ-ineq., $d(x, x') \leq d(x, x_n) + d(x_n, x')$ and for $\forall^\infty n$, $d(x, x_n) < \varepsilon/2$ and $d(x', x_n) < \varepsilon/2$, so $d(x, x') < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\square$

Obs. Any convergent sequence (x_n) is **bounded**, i.e. $\operatorname{diam}(\{x_0, x_1, \dots\}) < \infty \iff \exists$ ball $B \supseteq \{x_0, x_1, \dots\}$.

Proof. Let $x_n \to x$. For $\varepsilon := 1$, we know that $\operatorname{diam}(\{x_n, x_{n+1}, \dots\}) < 2$ $\forall^\infty n$, so $\operatorname{diam}(\{x_0, x_1, \dots, x_n, x_{n+1}, \dots\}) \le 2 + \max_{k=0}^{n} d(x_k, x_n) < \infty$, by $\triangle$-ineq. $\square$

Examples.

- In $\mathbb{R}$, $\frac{1}{n} \to 0$, $\frac{(-1)^n}{n} \to 0$, but $(-1)^n$ doesn't converge although is bdd, $x_n := n$ doesn't converge in $\mathbb{R}$.
- In $\mathbb{N}^{\mathbb{N}}$, for any $x \in \mathbb{N}^{\mathbb{N}}$, $x_n := (x|_n)12345\dots$, $x_n \to x$ because $\forall$ ball around x is of the form $[x|_N]$, so $\forall n \ge N$ $x_n \in [x|_N]$.



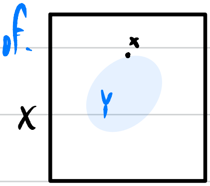
Obs. In $\mathbb{N}^{\mathbb{N}}$, $x_n \to x \iff \forall$ index $i \in \mathbb{N}$ $\forall^\infty n$ $x_n(i) = x(i)$.

- In $\mathbb{N}^{\mathbb{N}}$, $x_n := n000\dots$ doesn't converge.
- In $2^{\mathbb{N}}$, $x_{2k} := 0000\dots$, $x_{2k+1} := 1000\dots$, then (x_n) doesn't converge.

Prop (Closure via limits). For a metric space (X, d) & $Y \subseteq X$, $\forall x \in X$,

$x \in \overline{Y} \iff \exists$ sequence $(y_n) \subseteq Y$ s.t. $y_n \to x$.

Proof.



$\Rightarrow$. $\forall n$, $\exists\, y_n \in Y$ such that $y_n \in B_{\frac{1}{n}}(x)$. Thus, by def, $y_n \to x$ here $d(y_n, x) < \frac{1}{n} \to 0$.

$\Leftarrow$. $\forall$ neighborhood U of x, $\forall^\infty n$ $x_n \in U$, in particular, $Y \cap U \neq \emptyset$, so x is an adherent point to Y, thus $x \in \overline{Y}$. $\square$

Subsequences. A subsequence (x_{n_k}) of a sequence (x_n) is just another sequence (y_k) where $y_k = x_{n_k}$ for some strictly increasing sequence (n_k) of natural numbers. In particular, $n_k \geq k$.

Prop. For a sequence (x_n) in a metric space (X,d) and $x \in X$, TFAE:

(1) $x_n \to x$

(2) $\forall (x_{n_k}) \to x$.

false for ∞

(3) $\forall$ partition of (x_n) into $\boxed{\text{finitely many}}$ subsequences and each of them converge to x.

(4) $\forall$ subsequence (x_{n_k}) $\exists$ further subsequence $(x_{n_{k_l}}) \to x$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Trivial.

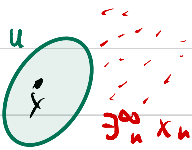
(3) $\Rightarrow$ (1). This follows from the fact that

$$\forall^\infty_{n_1} P_1(n_1) \wedge \forall^\infty_{n_2} P_2(n_2) \wedge \ldots \wedge \forall^\infty_{n_k} P_k(n_k)$$
$$\Longleftrightarrow \quad \forall^\infty_n \left(P_1(n) \wedge P_2(n) \wedge \ldots \wedge P_k(n)\right).$$

(Proof. Take the max of $N_1, N_2, \ldots, N_k$.) $\square$

(1) $\Rightarrow$ (4). Trivial.

(4) $\Rightarrow$ (1). We prove the contrapositive. Suppose (x_n) doesn't converge to x. Then $\exists$ neighbourhood U of x s.t. $\exists^\infty n \; x_n \notin U$. Then $\exists$ subsequence (comprised of these indices) (x_{n_k}) s.t. $\forall k \in \mathbb{N}$ $x_{n_k} \notin U$. By def, there is no subsequence of (x_{n_k}) that converges to x. $\square$



Characterization of closed via limits. Let (X, d) be a metric space and let $Y \subseteq X$. Then Y is closed iff (if and only iff) $\forall (y_n) \subseteq Y \; \forall x \in X$ if $y_n \to x$ then $x \in Y$.

Proof. $\Rightarrow$. Let $(y_n) \subseteq Y$ be s.t. $y_n \to x$. Then $\forall$ neighbourhood U of x, $\forall^\infty_n \; y_n \in U$ so $Y \cap U \neq \emptyset$, so $x \in \overline{Y} = Y$ because Y is closed.

$\Leftarrow$. We need to show that $\overline{Y} = Y$. Let $x \in \overline{Y}$.

Then, by what we proved above, $\exists (y_n) \subseteq Y$ s.t. $y_n \to x$. But then $x \in Y$ by our assumption. $\square$

Cauchy sequences. A sequence (x_n) in a metric space (X, d) is called **Cauchy** if the diameter of the tail of the sequence converges to 0, i.e.

$$\lim_n \operatorname{diam}(\{x_n, x_{n+1}, \dots\}) = 0.$$

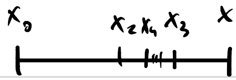
In other words, $\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N \ d(x_n, x_m) < \varepsilon$.
In particular, $\operatorname{diam}(\{x_0, x_1, x_2, \dots\}) < \infty$, so Cauchy sequences are bounded.

Examples.

- Every convergent sequence is Cauchy.
  Proof. By the triangle inequality.
  $x_n \to x$ then
  $$d(x_n, x_m) \leq \underbrace{d(x_n, x)}_{\to 0} + \underbrace{d(x_m, x)}_{\to 0} \text{ when } (n,m) \to (\infty, \infty).$$ $\square$

- A sequence (x_n) is called **contractive** if $\exists \alpha \in (0,1)$ s.t. $\forall n \ d(x_{n+2}, x_{n+1}) \leq \alpha \cdot d(x_{n+1}, x_n)$.

Example. $x_0, x_1 \in \mathbb{R}$, $x_0 < x_1$



$x_{n+2} := \frac{1}{2}(x_{n+1} + x_n)$, i.e. each term is the average of the previous two terms.

$$d(x_{n+2}, x_{n+1}) = |x_{n+2} - x_{n+1}| = \left|\tfrac{1}{2}x_{n+1} + \tfrac{1}{2}x_n - x_{n+1}\right| = \tfrac{1}{2}|x_n - x_{n+1}| = \tfrac{1}{2}\, d(x_n, x_{n+1}).$$

so (x_n) is contractive.

Prop. Contractive sequences are Cauchy.

Proof. $\forall n, m \in \mathbb{N}$,

$$d(x_n, x_{n+m}) \underset{\Delta\text{-ineq.}}{\le} d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+m-1}, x_{n+m})$$

$$= d(x_n, x_{n+1}) + \alpha d(x_n, x_{n+1}) + \alpha^2 d(x_n, x_{n+1}) + \ldots + \alpha^{m-1} d(x_n, x_{n+1})$$

$$= d(x_n, x_{n+1}) \cdot (1 + \alpha + \alpha^2 + \ldots + \alpha^{m-1})$$

$$= d(x_n, x_{n+1}) \cdot \frac{1 - \alpha^m}{1 - \alpha}$$

$$\le \alpha^n d(x_0, x_1) \cdot \frac{1}{1 - \alpha} \to 0 \text{ as } n \to \infty. \qquad \square$$

Counterexample for $d(x_{n+2}, x_{n+1}) < d(x_{n+1}, x_n)$. In $\mathbb{R}$,

$x_n := \sum_{k=1}^{n} \frac{1}{k}$, then $d(x_{n+2}, x_{n+1}) = \frac{1}{n+2} < \frac{1}{n+1} = d(x_{n+1}, x_n)$

but (x_n) doesn't converge.